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Kolmogorov's Type Criteria for Spaces of Compact Operators

GRZEGORZ LEWICKI

Department of Mathematics, Jagiellonian University, Reymonta 4, 30-059 Krakow, Poland

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The aim of this paper is to prove various Kolmogorov's type criteria for spaces of compact operators. We also present the results concerning strongly unique best approximation. In particular we generalize some well known theorems from the theory of minimal projections. As an application, we characterize SUBA projections onto hyperplanes in l^n_∞ and estimate the strong strong unicity constant in this case. © 1991 Academic Press, Inc.

0. INTRODUCTION

Let $C(T)$ denote the space of all continuous, complex valued functions defined on a compact set T with the supremum norm $\| \ \|$. For $f \in C(T)$ and $V \subset C(T)$ put

$$P_V(f) = \{v \in V: \|f - v\| = \operatorname{dist}(f, V)\}.$$

If V is a linear subspace of $C(T)$ then the classical Kolmogorov's criterion reads as

$$\begin{aligned} &v \in P_V(f) \quad \text{if and only if for every} \quad w \in V \\ &\inf\{\operatorname{re}((f(t) - v(t)) \cdot \overline{w(t)}): \ t \in C(f - v)\} \leqslant 0, \quad \text{where} \\ &C(f - v) = \{t \in T: |f(t) - v(t)| = \|f - v\|\}. \end{aligned} \tag{0.1}$$

The above characterization of best approximants can be extended to the case of an arbitrary Banach space. Namely, let W be a Banach space over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) and let $S(W^*)$ denote the unit sphere in the space W^*. For $w \in W$ put

$$E(w) = \{f \in \operatorname{ext} S(W^*): f(w) = \|w\|\} \tag{0.2}$$

and let for $V \subset W$

$$P_V(w) = \{v \in V: \|w - v\| = \operatorname{dist}(w, V)\}. \tag{0.3}$$

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Then we have

THEOREM 0.1 (see [2]). *For every $V \subset W$ the following conditions are equivalent*:

$$V \text{ is a sun (i.e., for every } w \in W,\ v \in P_V(w), \text{ and } t \geqslant 0,\ v \in P_V(v + t(w-v)); \tag{0.4}$$

$$\text{for every } w \in W,\ v \in P_V(w) \text{ if and only if for every } u \in V \text{ there exists } f \in E(w-v) \text{ such that } \operatorname{re}(f(u-v)) \leqslant 0. \tag{0.5}$$

The similar result can be proved in the case of strong unicity. In order to present it, let us recall that an element $v \in V$ is called the strongly unique best approximation (briefly SUBA) to $w \in W$ if and only if there exists $r > 0$ such that for every $u \in V$

$$\|w-u\| \geqslant \|w-v\| + r\,\|u-v\|. \tag{0.6}$$

In [19, Theorem 2.1, p. 885] the following was shown:

THEOREM 0.2. *Let $w \in W \backslash V$ and let V be a starlike set with respect to $v \in V$. Then the following statements are equivalent*:

$$v \text{ is a SUBA to } w \in W \text{ with a constant } r > 0 \tag{0.7}$$

$$\text{for every } u \in V\ \operatorname{re}(f(u-v)) \leqslant -r\,\|u-v\| \text{ for some } f \in E(w-v). \tag{0.8}$$

It is clear that each convex set is a sun and a starlike set with respect to each of its points, so the results presented above may be treated as the generalizations of Kolmogorov's criterion. However, in general, applications of them seem to be limited, because in many cases the form of the points from the set ext $S(W^*)$ is unknown.

In this paper, applying the previously mentioned results, we prove some criteria (Theorems 2.2, 2.3, 3.2, 3.5, 4.1, 4.4) in the case $W = \mathscr{K}(X, Y)$, where $\mathscr{K}(X, Y)$ denotes the space of all compact operators going from a normed space X to a Banach space Y. These characterizations are expressed in terms of the set ext $S(Y^*)$, which is more convenient for applications. In particular, we generalize some classical results from the theory of minimal projections (see Theorems 3.4, 3.5, 4.1, 4.4). On the other hand Theorem 2.5 illustrates how to apply Theorems 2.2 and 2.3 in concrete cases.

Now we briefly describe the contents of the paper. Section 1 contains notions, terminology, and preliminary results. In Section 2 we discuss the general case of spaces of compact operators. In Section 3 we specialize our results to compact operators from the space $\mathcal{K}(C(T), C(T))$ with discrete support. Section 4 deals with the case of sequence spaces $c_0(T)$, $l_1(T)$.

1. Preliminaries

During this paper, for a normed space X, we denote by $S(X)$ the unit sphere in X and by ext $S(X)$ the set of all its extremal points. Given a normed space X and a Banach space Y, both over the same field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), we write $\mathcal{K}(X, Y)$ for the space of all compact operators going from X into Y. The symbol $\mathcal{L}_e(X^*, Y)$ stands for the space of all weak*-weakly continuous compact operators from X^* into Y endowed with the operator norm.

Applying Goldstine's Theorem we may prove the following

PROPOSITION 1.1 (see [11, Example (0.2)]). *The space $\mathcal{K}(X, Y)$ is linearly isometric with the space $\mathcal{L}_e(X^{**}, Y)$. This isometry, denoted by *, is given by*

$$L^*f = \lim_{\beta} Lx_\beta, \tag{1.1}$$

where $L \in \mathcal{K}(X, Y)$, $f \in X^{**}$, and a net $(x_\beta) \subset X$ is so chosen that $x_\beta \to f$ weak in X^{**}.*

The next theorem plays a crucial role in our investigations.

THEOREM 1.2 (see [11, Theorem 2.2(a)]).

$$\text{ext } S(\mathcal{L}_e^*(X^*, Y)) \subset \text{ext } S(X^*) \otimes \text{ext } S(Y^*), \tag{1.2}$$

where $(x^* \otimes y^*)(L) = y^*(Lx^*)$ for $x^* \in X^*$, $y^* \in Y^*$, and $L \in \mathcal{L}_e(X^*, Y)$.

By Proposition 1.1, we immediately obtain

COROLLARY 1.3. *For each $f \in \text{ext } S(\mathcal{K}^*(X, Y))$ there exist $y^* \in \text{ext } S(Y^*)$ and $x^{**} \in \text{ext } S(X^{**})$ such that $f(L) = (x^{**} \otimes y^*)(L^*)$ for every $L \in \mathcal{K}(X, Y)$.*

Remark 1.4. If $L \in \mathcal{K}(X, Y)$ is a finite dimensional operator then

$$L^*f = \sum_{i=1}^{n} f(x_i^*) \cdot y_i \qquad \text{for} \quad f \in X^{**},$$

where $L = \sum_{i=1}^{n} x_i^*(\cdot) \cdot y_i$.

Now, following [7], we recall a notion of the support of linear operator.

DEFINITION 1.5. For $X = C(T)$ and $L \in \mathscr{L}(X, Y)$ let us set

$$\mathscr{F} = \{F \subset T\colon F \text{ is closed and for every } x \in C(T),\ Lx = 0 \text{ if } x|_F = 0\}.$$

The smallest, in the sense of inclusion, set $F_0 \in \mathscr{F}$ is called the support of the operator L (we write $\operatorname{supp}(L)$ for brevity).

The existence of such a set for every $L \in \mathscr{L}(X, Y)$ was proved in [7, p. 64]. If the $\operatorname{supp}(L)$ is finite, then the operator L is called discrete. The set of all discrete operators going from X into Y is denoted by $\mathscr{D}(X, Y)$ ($\mathscr{D}(X)$ if $X = Y$).

Now we present the notions and the terminology concerning sequence spaces.

Given an arbitrary set T by $c_0(T)$, written c_0 for brevity, we denote the space of all functions $x\colon T \to \mathbb{K}$ such that the set $\{t\colon |x(t)| > \varepsilon\}$ is finite for all $\varepsilon > 0$. The norm in c_0 is $\|x\|_\infty = \sup\{|x(t)|\colon t \in T\}$. The space $l_1(T)$ consists of all functions $x\colon T \to \mathbb{K}$ which are zero except on a countable set in T for which $\|x\|_1 = \sum_{t \in T} |x(t)| < \infty$. It is well known that the conjugate space of c_0 can be isometrically identified with $l_1(T)$ (written l_1 for brevity) and the conjugate space of l_1 with l_∞, where

$$l_\infty = \{x\colon T \to \mathbb{K}\colon \sup\{|x(t)|\colon t \in T\} < +\infty\}. \tag{1.3}$$

We note that

$$\operatorname{ext} S(l_1) = \{\alpha \cdot f_t\colon t \in T, \alpha \in \mathbb{K}, |\alpha| = 1\}, \tag{1.4}$$

where

$$f_t(s) = \begin{cases} 0, & s \neq t \\ 1, & s = t \end{cases}$$

and

$$\operatorname{ext} S(l_\infty) = \{f\colon T \to \mathbb{K}\colon |f(t)| = 1 \text{ for every } t \in T\}. \tag{1.5}$$

By [12, Theorem 18, p. 274], the set $\operatorname{ext} S(l_\infty^*)$ has the following representation:

$$\operatorname{ext} S(l_\infty^*) = \operatorname{cl}\{\hat{t}\colon t \in T\}, \tag{1.6}$$

where $\hat{t}(f) = f(t)$ for every $f \in l_\infty$ and the closure is taken with respect to the weak* topology in l_∞^*.

At the end of this section we recall the notion of projection. If Y is a linear subspace of a Banach space X, then a projection of X onto Y is a bounded linear map $P: X \to Y$ such that $Py = y$ for every $y \in Y$. The set of all projections going from X onto Y is denoted by $\mathscr{P}(X, Y)$. A projection $P_0 \in \mathscr{P}(X, Y)$ of minimal norm in $\mathscr{P}(X, Y)$ is called a minimal projection. Many applications of projections occur in numerical analysis and approximation theory, for Px can be regarded as an approximation to x in Y. The quality of this approximation relative to the best approximation is governed by the inequality

$$\|x - Px\| \leqslant \|I - P\| \cdot \operatorname{dist}(x, Y) \leqslant (1 + \|P\|) \cdot \operatorname{dist}(x, Y) \quad \text{for} \quad x \in X. \tag{1.7}$$

If $X = C(T)$, by $\mathscr{P}(X, Y, F)$ we denote the set of all projections such that $\operatorname{supp}(P) \subset F$.

If $Y \subset X$ is an n-dimensional subspace we write $I(X, Y)$ for the set of all interpolating projections, i.e.,

$$P \in I(X, Y) \text{ if and only if } P = \sum_{i=1}^{n} \hat{t}_i(\cdot) \cdot y_i, \tag{1.8}$$

where $t_i \in T$, $y_i \in Y$ for $i = 1, ..., n$.

For a more complete list of information about projections the reader is referred to [1, 4, 5, 7–10, 13–15, 17, 18].

2. General Case

We start with the following

Lemma 2.1. *Let X be a normed space and let Y be a Banach space, both over the same field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). For $L \in \mathscr{K}(X, Y)$ put*

$$\operatorname{crit}(L) = \{f \in \operatorname{ext} S(Y^*) : \|f \circ L\| = \|L\|\}. \tag{2.1}$$

Then the set $\operatorname{crit}(L)$ *is nonvoid for every $L \in \mathscr{K}(X, Y)$.*

Proof. Fix $L \in \mathscr{K}(X, Y)$ and consider the function $\phi(f) = \|f \circ L\|$ for $f \in S(Y^*)$. We show that ϕ is weak* continuous on $S(Y^*)$. By the compactness of L the space $L(X)$ is separable and since $f \circ L = f|_{L(X)} \circ L$ we may restrict ourselves to the case when Y is separable. Following [12, Theorem 1, p. 426], the space $S(Y^*)$ with the weak* topology is metrizable in this case. Now suppose on the contrary that $\{f_n\} \subset S(Y^*)$ tends weak* to $f \in S(Y^*)$ and $\phi(f_n - f) \geqslant \varepsilon > 0$. Then $(f_n - f)(Lx_n) > \varepsilon/2$ for some

$\{x_n\} \subset S(X)$. By the compactness of L we may assume $\|Lx_n - y\| \to 0$ for some $y \in Y$. We note that

$$\begin{aligned} |(f_n - f)(Lx_n)| &\leqslant |(f_n - f)(Lx_n - y)| + |(f_n - f)(y)| \\ &\leqslant 2 \cdot \|Lx_n - y\| + |(f_n - f)(y)| \\ &\leqslant \frac{\varepsilon}{2} \quad \text{for} \quad n \geqslant n_0; \end{aligned}$$

then we have a contradiction. Applying the Banach–Alaoglu and the Krein–Milman Theorems we complete the proof.

Now we prove the main result of this section.

THEOREM 2.2. *Let X, Y be such as in Lemma 2.1. Assume $\mathscr{V} \subset \mathscr{K}(X, Y)$ is a convex set. Let $K \in \mathscr{K}(X, Y)$ and $V \in \mathscr{V}$. Then we have:*

(a) *$V \in P_{\mathscr{V}}(K)$ (see (0.3)) if and only if for every $U \in \mathscr{V}$ there exists $y^* \in \operatorname{crit}(K - V)$ such that $\|\operatorname{re}(y^* \circ (K - U))\| \geqslant \|K - V\|$.*

(b) *V is a SUBA to K in $\mathscr{V}$ with a constant $r > 0$ if and only if for every $U \in \mathscr{V}$ there exists $y^* \in \operatorname{crit}(K - V)$ such that*

$$\|\operatorname{re}(y^* \circ (K - U))\| \geqslant \|K - V\| + r \cdot \|K - U\|.$$

Proof. (a) Fix $U \in \mathscr{V}$. Since $\|\operatorname{re}(y^* \circ (K - U))\| \geqslant \|K - V\|$ for some $y^* \in \operatorname{crit}(K - V)$ $V \in P_{\mathscr{V}}(K)$.

To prove the converse, assume that there exists $U \in \mathscr{V}$ such that $\|\operatorname{re}(y^* \circ (K - U))\| < \|K - V\|$ for every $y^* \in \operatorname{crit}(K - V)$. Take an arbitrary $f \in E(K - V)$ (see (0.2)). By Theorem 1.2 and Corollary 1.3, $f = x^{**} \otimes y^*$ for some $x^{**} \in \operatorname{ext} S(X^{**})$ and $y^* \in \operatorname{ext} S(Y^*)$. Following Goldstine's Theorem select a net $(x_\beta) \subset S(X)$ such that x_β tends to x^{**} weak* in X^{**}. Since, by (1.1), $\operatorname{re}(y^* \circ (K - V) x_\beta)$ tends to $\operatorname{re}(y^* \circ (K - V)^* x^{**}) = \operatorname{re}((x^{**} \otimes y^*)(K - V)) = \operatorname{re}((f(K - V)), y^* \in \operatorname{crit}(K - V)$. Hence we have

$$\begin{aligned} \operatorname{re}(f(U - V)) &= \operatorname{re}(f(K - V)) - \operatorname{re}(f(K - U)) \\ &= \|K - V\| - \operatorname{re}(y^*((K - U)^* x^{**})) \\ &= \|K - V\| - \lim_\beta \operatorname{re}(y^*((K - U) x_\beta)) \\ &\geqslant \|K - V\| - \|\operatorname{re}(y^* \circ (K - U))\| > 0. \end{aligned}$$

Following Theorem (0.1), $V \notin P_{\mathscr{V}}(K)$, a contradiction.

By the same reasoning, applying Theorem 0.2, we can prove part (b).

For X being a reflexive space we can show the following

THEOREM 2.3. *Assume X is a reflexive space and let Y, $\mathscr{V}$, K, V be such as in Theorem 2.2. For $y^* \in \operatorname{crit}(K-V)$ put*

$$A_{y^*} = \{x \in S(X) \colon y^*((K-V)x) = \|K-V\|\}. \tag{2.2}$$

Then we have:

(a) *$V \in P_{\mathscr{V}}(K)$ if and only if for every $U \in \mathscr{V}$ there exists $y^* \in \operatorname{crit}(K-V)$ such that $\inf\{\operatorname{re}(y^*(U-V)x) \colon x \in A_{y^*}\} \leqslant 0$.*

(b) *V is a SUBA to K in $\mathscr{V}$ with a constant $r > 0$ if and only if for every $U \in \mathscr{V}$ there exists $y^* \in \operatorname{crit}(K-V)$ with*

$$\inf\{\operatorname{re}(y^*(U-V)x) \colon x \in A_{y^*}\} \leqslant -r \cdot \|U-V\|.$$

Proof. Assume $V \notin P_{\mathscr{V}}(K)$. Then $\|K-U\| < \|K-V\|$ for some $U \in \mathscr{V}$. Take an arbitrary $y^* \in \operatorname{crit}(K-V)$ and $x \in A_{y^*}$. Compute

$$\begin{aligned} \operatorname{re}(y^*(U-V)x) &= \operatorname{re}(y^*(K-V)x) - \operatorname{re}(y^*(K-U)x) \\ &\geqslant \|K-V\| - \|K-U\| > 0 \end{aligned}$$

and consequently $\inf\{\operatorname{re}(y^*(U-V)x) \colon x \in A_{y^*}\} > 0$.

To prove the converse, suppose that $\inf\{\operatorname{re}(y^*(U-V)x) \colon x \in A_{y^*}\} > 0$ for every $y^* \in \operatorname{crit}(K-V)$ (the set A_{y^*} is nonvoid by the reflexivity of X). Take an arbitrary $f \in E(K-V)$. In view of Theorem 1.2, $f = x^{**} \otimes y^*$ for some $y^* \in \operatorname{ext} S(Y^*)$ and $x^{**} \in \operatorname{ext} S(X^{**})$. Since X is reflexive, $x^{**} = x$, for some $x \in S(X)$. It is clear that $y^* \in \operatorname{crit}(K-V)$ and $x \in A_{y^*}$. Consequently $\operatorname{re}(f(U-V)) > 0$ and, by Theorem 0.1, $V \notin P_{\mathscr{V}}(K)$.

Applying Theorem 0.2, by the same reasoning we can prove part (b).

Remark 2.4. If X is an arbitrary normed space it may occur that the set A_{y^*} is empty. Take, for example, $X = C_0^{2\pi}$, the space of all real, 2π periodic continuous functions, and let Y_n be the space of all trigonometric polynomials of degree $\leqslant n$. Put $\mathscr{V} = \mathscr{P}(X, Y_n)$, the space of all projections going from X onto Y_n. It is well known (see, e.g., [3, p. 212]) that the classical Fourier projection F_n is minimal among all projections, which means $F_n \in P_{\mathscr{V}}(0)$. Following [17, Lemma 4.1], F_n cannot attain its norm in any point of $S(X)$. Consequently for every $y^* \in \operatorname{crit}(F_n)$ the set A_{y^*} is empty.

Now we apply Theorems 2.2(b) and 2.3(b) in the case when $\mathscr{V} = \mathscr{P}(l_\infty^n, Y)$, $K = 0$ ($l_\infty^n = l_\infty(\{1, \ldots, n\})$ and Y is a hyperplane in l_∞^n). In other words we show when a minimal projection $P_0 \in \mathscr{P}(l_\infty^n, Y)$ satisfies the inequality

$$\|P\| \geqslant \|P_0\| + r \cdot \|P - P_0\| \qquad \text{for every} \quad P \in \mathscr{P}(l_\infty^n, Y), \tag{2.3}$$

where the constant $r > 0$ is independent of $P \in \mathscr{P}(l_\infty^n, Y)$.

THEOREM 2.5. *Let* $Y \subset l_\infty^n$ *be a hyperplane; i.e.,* $Y = \ker f$ *for some* $f = (f_1, ..., f_n) \in l_1^n = l_1(\{1, ..., n\})$, $\|f\|_1 = 1$. *Assume* $P_0 \in \mathscr{P}(l_\infty^n, Y)$ *is a minimal projection. Then we have*:

(a) *If* $\|P_0\| = 1$, *then* P_0 *satisfies* (2.3) *if and only if* $|f_i| \geq 1/2$ *for exactly one index* $i \in \{1, ..., n\}$. *The constant* $r = \min\{1 - 2 \cdot |f_j|; j \neq i\}$ *is the best possible.*

(b) *In the real case, if* $\|P_0\| > 1$ *then* P_0 *satisfies* (2.3) *if and only if* $0 < |f_i| < 1/2$ *for* $i = 1, ..., n$.

Moreover, the constant

$$r = \min\{\max\{(1 - 2 \cdot |f_i|) \cdot y_i: i = 1, ..., n\}: y \in S(Y)\}$$

is the best possible and there holds an estimation

$$r \geq (1 - 2 \cdot |f_j|) \cdot |f_i|/(1 - |f_i|),$$

where $|f_j| = \max\{|f_k|: k = 1, ..., n\}$ *and*

$$|f_i| = \min\{|f_k|: k = 1, ..., n\}.$$

Proof. (a) Assume that $|f_i| \geq 1/2$ for exactly one index $i \in \{1, ..., n\}$. Following [1, Lemma 1] each $P \in \mathscr{P}(l_\infty^n, Y)$ has the representation

$$Px = x - f(x) \cdot y^P = P_{f, y^P} \qquad \text{for} \quad x \in l_\infty^n, \tag{2.4}$$

where $y^P \in l_\infty^n$ satisfies $\sum_{i=1}^n f_i \cdot y_i^P = 1$. Hence $P - P_0 = f(\cdot) \cdot (y^{P_0} - y^P)$ for every $P \in \mathscr{P}(l_\infty^n, Y)$. It is clear that $\|P - P_0\| = \|y^{P_0} - y^P\|_\infty$. Since $|f_i| \geq 1/2$, $\|y^{P_0} - y^P\|_\infty = |y_j^P - y_j^{P_0}|$ for some $j \neq i$. By [1, Theorem 1], $y_i^{P_0} = 1/f_i$ and $y_j^{P_0} = 0$ for $j \neq i$. Consequently $\|P - P_0\| = |y_j^P|$ for some $j \neq i$. By Lemma 2 of [1], we note that

$$\begin{aligned}\|P\| &\geq \|(x \to x_j) \circ P\| = |1 - f_j \cdot y_j^P| + |y_j^P| \cdot (1 - |f_j|) \\ &\geq 1 + |y_j^P| \cdot (1 - 2 \cdot |f_j|) \\ &\geq \|P_0\| + \min\{1 - 2 \cdot |f_k|: k \neq i\} \cdot \|P - P_0\|,\end{aligned}$$

which gives the result.

Now we will show that the constant $r = \min\{1 - 2 \cdot |f_j|: j \neq i\}$ is the best possible. Since $\|P_{f, y}\| = \|P_{|f|, \bar{y}}\|$ for every $f \in l_1^n$ and $y \in \ker(f)$ ($\bar{y}_i = y_i$ if $f_i = 0$ and $\bar{y}_i = f_i/|f_i| \cdot y_i$ in the other case) we may assume $f \geq 0$. Set $y_k = 0$, if $k \neq i$ and $k \neq j$, $y_i = -f_j/f_i$, $y_j = 1$ and let $y = (y_1, ..., y_n)$ (the index j is so chosen that $f_j = \max\{f_k: k \neq i\}$). Let $P = P_0 - f(\cdot) \cdot y$. By Theorem 2.2(b) and (1.4), it is enough to show that

$$\|(x \to x_k) \circ P\| < 1 + r_1 \cdot \|P - P_0\| \qquad \text{for every} \quad r_1 > r \text{ and } k = 1, ..., n.$$

At first we note that $\|P-P_0\| = \|y\|_\infty = 1$. Following [1, Lemma 2],

$$\|(x \to x_k) \circ P\| = |1 - f_k \cdot y_k^P| + |y_k^P| \cdot |1 - f_k| \qquad \text{for} \quad k = 1, ..., n.$$

So if $k = i$, then

$$\begin{aligned}\|(x \to x_k) \circ P\| &= |1 - f_i \cdot (y_i + 1/f_i)| + |y_i + 1/f_i| \cdot |1 - f_i| \\ &= 1/f_i - 1 + y_i \cdot (1 - 2 \cdot f_i) \\ &= 1/f_i - 1 + f_j \cdot (2 \cdot f_i - 1)/f_i \\ &\leqslant 1/f_i - 1 + (1 - f_i) \cdot (2 \cdot f_i - 1)/f_i \\ &= 2 \cdot (1 - f_i) \leqslant 1 < 1 + r_1 \cdot \|P - P_0\|.\end{aligned}$$

If $k \neq i$ and $k \neq j$, then $y_k^P = y_k = 0$. Hence

$$\|(x \to x_k) \circ P\| = 1 < 1 + r_1 \cdot \|P - P_0\|.$$

If $k = j$, then

$$\|(x \to x_k) \circ P\| = 2 - 2 \cdot f_j = 1 + r \cdot \|P - P_0\| < 1 + r_1 \cdot \|P - P_0\|.$$

Applying Theorem 2.2(b), we complete the proof of part (a).

(b) As in the previous case we may assume $f_i \geqslant 0$ for $i = 1, ..., n$. Let us define a function ϕ: $S(Y) \to \mathbb{R}$ by the formula

$$\phi(y) = \min\{(2 \cdot f_i - 1) \cdot y_i : i = 1, ..., n\}.$$

Since $f_i > 0$ for $i = 1, ..., n$, $\phi(y) < 0$ for every $y \in S(Y)$. Hence, by the argument of compactness and continuity of ϕ, the constant $\gamma = \max\{\phi(y): y \in S(Y)\}$ is negative. We show that P_0 is a SUBA to 0 in $\mathscr{P}(l_\infty^n, Y)$ with $r = -\gamma$. To do this, following Theorem 2.3(b), (1.4), and Theorem (10) of [7], it is enough to prove that for every $P \in \mathscr{P}(l_\infty^n, Y)$ there exists $i \in \{1, ..., n\}$ with

$$\inf\{((P - P_0)x)_i : x \in A_i\} \leqslant -r \cdot \|P - P_0\| \tag{2.5}$$

(we write A_i instead of $A_{x \to x_i}$).

By (2.4), $\|P - P_0\| = \|y^P - y^{P_0}\|_\infty$. Set $y = (y^P - y^{P_0})/\|y^P - y^{P_0}\|_\infty$ (if $y^P = y^{P_0}$ the inequality (2.5) is satisfied). Select $i \in \{1, ..., n\}$ with $\phi(y) = (2 \cdot f_i - 1) \cdot y_i$. Following [1, Theorem 2 and Lemma 2], $x \in A_i$ if and only if $x_j = -\text{sgn}(f_j) = -1$ for $j \neq i$ and $x_i = \text{sgn}(1 - f_i \cdot y_i^{P_0}) = 1$. Hence, for $x \in A_i$,

$$\begin{aligned}((P - P_0)x)_i &= f(x) \cdot \|y^P - y^{P_0}\|_\infty \cdot y_i = (2 \cdot f_i - 1) \cdot y_i \cdot \|y^P - y^{P_0}\|_\infty \\ &\leqslant -r \cdot \|y^P - y^{P_0}\|\end{aligned}$$

which by Theorem 2.3(b) gives the desired result.

Now we will show that $r \geqslant (1-2\cdot f_j)\cdot f_i/(1-f_i)$, where $f_j = \max\{f_k: k=1, ..., n\}$ and $f_i = \min\{f_k: k=1, ..., n\}$. To do this, take $y \in S(Y)$. If $y_k = 1$ for some $k \in \{1, ..., n\}$, then

$$\phi(y) \leqslant 2\cdot f_k - 1 \leqslant 2\cdot f_j - 1 \leqslant (2\cdot f_j - 1)\cdot f_i/(1-f_i),$$

since $f_j < 1/2$ and $f_i < 1/2$.

In the opposite case $y_k = -1$ for some $k \in \{1, ..., n\}$ and an easy calculation shows that $y_l \geqslant f_i/(1-f_i)$ for some $l \in \{1, ..., n\}$. We note that

$$\phi(y) \leqslant (2\cdot f_l - 1)\cdot y_l \leqslant (2\cdot f_l - 1)\cdot f_i/(1-f_i) \leqslant (2\cdot f_j - 1)\cdot f_i/(1-f_i),$$

since $f_l < 1/2$ and $f_j \geqslant f_l$.

Hence $\gamma \leqslant (2\cdot f_j - 1)\cdot f_i/(1-f_i)$ and consequently

$$r \geqslant (1-2\cdot f_j)\cdot f_i/(1-f_i).$$

To prove that the constant r is the best possible, take $r_1 > r$, choose $y \in S(Y)$ with $\phi(y) > -r_1$, and define $P \in \mathscr{P}(l^n_\infty, Y)$ by $P = P_0 + f(\cdot)\cdot y$. For $l \in \{1, ..., n\}$ and $x \in A_l$ we have

$$((P-P_0)x)_l = f(x)\cdot y_l = (2\cdot f_l - 1)\cdot y_l \geqslant \phi(y) > -r_1 = -r_1 \cdot \|P-P_0\|.$$

Since $A_i = -A_{x \to -x_i}$, by Theorem 2.3(b), the proof of part (b) is fully completed.

Remark 2.6. In the complex case Theorem 2.5(b) does not hold.

Proof. As in the proof of Theorem 2.5(b) we may assume $f \geqslant 0$. It is easy to show that the projection P_0 considered in Theorem 2.5(b) is minimal in the complex case. By (1.4) and easy calculation $A_i = \alpha \cdot A_{x \to \alpha\cdot x_i}$ for every $\alpha \in \mathbb{C}$, $|\alpha| = 1$. Hence we may restrict ourselves to the case $\alpha = 1$.

Take $w \in \mathbb{R}^n \cap S(Y)$ and let $y = 0 + i\cdot w$. For $L = f(\cdot)\cdot y$, $j = 1, ..., n$ and $x \in A_j$ we have $\mathrm{re}(Lx)_j = \mathrm{re}(f(x)\cdot y_j) = (2\cdot f_j - 1)\cdot \mathrm{re}(y_j) = 0 > -r\cdot\|y\|$ for every $r > 0$.

Hence, by Theorem 2.3(b), P_0 does not satisfy (2.3) with any constant $r > 0$.

However, adopting the reasoning from [1, Theorem 2], we can show that the conditions given in Theorem 2.5(b) are equivalent to the uniqueness of minimal projection in the complex case.

3. Criteria for the Space $\mathscr{K}(C(T))$

During this section $X = C(T)$, i.e., the space of all continuous, complex valued functions defined on a compact set T with the supremum norm. For

$F \subset T$, by $\mathcal{K}_F(X)$ ($\mathcal{K}(X)$ if $F = T$) we denote the space of all compact operators going from X to X with supports (see Definition 1.5) contained in F. For $t \in T$ the symbol $\hat{t}$ stands for the evaluation functional.

We start with the following

LEMMA 3.1. *Assume that* $V \in \mathcal{K}(X)\{0\}$ *and let* $\operatorname{card}(\operatorname{supp}(V)) < \infty$, *i.e.*, $V \in \mathcal{D}(X)$. *For* $\hat{t} \in \operatorname{crit}(V)$ (*see* 2.1) *put*

$$A_t = \{x \in S(X) \colon (Vx)t = \|V\|\}. \tag{3.1}$$

Then for every $\hat{t} \in \operatorname{crit}(V)$ *and every* $\{x_n\} \subset S(X)$ *with* $(Vx_n)t \to \|V\|$, *there exists* $\{z_n\} \subset A_t$ *with* $\|z_n - x_n\| \to 0$ *as* $n \to \infty$.

Proof. Since $V \in \mathcal{D}(X)$, $V = \sum_{i=1}^k \hat{t}_i(\cdot) \cdot y_i$, where $y_i \in X$, $t_i \in T$ for $i = 1, \ldots, k$. By the Tietze–Urysohn Theorem $\|V\| = \|\sum_{i=1}^k |y_i|\,\|$. Fix $\hat{t} \in \operatorname{crit}(V)$, $\{x_n\} \subset S(X)$ with $(Vx_n)t \to \|V\|$, and let $A = \{i \in \{1, \ldots, k\} \colon y_i(t) \neq 0\}$. Ar first we will show that $x_n(t_i) \to \overline{y_i(t)}/|y_i(t)| = \operatorname{sgn}(y_i(t))$ for $i \in A$. Since $\sum_{i=1}^k |y_i(t)| = \sum_{i \in A} \operatorname{sgn}(y_i(t)) \cdot y_i(t)$, $|x_n(t_i)| \to 1$ for each $i \in A$. Assume that for some $i_0 \in A$ there exists a subsequence (x_{n_k}) with

$$|\operatorname{sgn}(y_{i_0}(t)) - x_{n_k}(t_{i_0})/|x_{n_k}(t_{i_0})|\,| \geqslant d > 0 \qquad \text{for} \quad k \geqslant k_0.$$

By the uniform convexity of $\mathbb{C}$ over $\mathbb{R}$,

$$|\tfrac{1}{2} \cdot (\operatorname{sgn}(y_{i_0}(t)) + x_{n_k}(t_{i_0})/|x_{n_k}(t_{i_0})|)| \leqslant 1 - \delta \qquad \text{for some} \quad \delta > 0.$$

Compute

$$\left|\frac{1}{2} \cdot \left(\sum_{i \in A} |y_i(t)| + \sum_{i \in A} ((x_{n_k}(t_i)/|x_{n_k}(t_i)|) \cdot y_i(t))\right)\right|$$
$$\leqslant \sum_{i \in A \setminus \{i_0\}} |y_i(t)| + \left|\frac{1}{2} \cdot (\operatorname{sgn}(y_{i_0}(t)) + x_{n_k}(t_{i_0})/|x_{n_k}(t_{i_0})|)\right| \cdot |y_{i_0}(t)|$$
$$\leqslant \sum_{i \in A \setminus \{i_0\}} |y_i(t)| + (1 - \delta) \cdot |y_{i_0}(t)| < \|V\|.$$

But, passing to the subsequence if necessary, $\sum_{i \in A} (x_{n_k}(t_i)/|x_{n_k}(t_i)|) \cdot y_i(t)$ tends to $\|V\|$ as $k \to \infty$; then we have a contradiction.

Now we construct the sequence (z_n). For each $n \in N$ let us set

$$\varepsilon_n = \max\{|x_n(t_i) - \operatorname{sgn}(y_i(t))| \colon i \in A\}.$$

Fix $n \in N$ and for every $i \in A$ select an open neighbourhood U_i of t_i such that $\bar{U}_i \cap \bar{U}_j = \emptyset$ for $i \neq j$ and $|x_n(s) - x_n(t_i)| \leqslant \varepsilon_n$ for $s \in \bar{U}_i$, $i \in A$. Fix $i \in A$. An easy calculation shows that for every $s \in \bar{U}_i$

$$\begin{aligned}\operatorname{re}(x_n(s)) \in [&\operatorname{re}(\operatorname{sgn}(y_i(t))) - 2 \cdot \varepsilon_n, \operatorname{re}(\operatorname{sgn}(y_i(t))) + 2 \cdot \varepsilon_n] \\ &\cap [-1, 1] = [B, C]\end{aligned}$$

and

$$\begin{aligned}\operatorname{im}(x_n(s)) \in [&\operatorname{im}(\operatorname{sgn}(y_i(t))) - 2 \cdot \varepsilon_n, \operatorname{im}(\operatorname{sgn}(y_i(t))) + 2 \cdot \varepsilon_n] \\ &\cap [-1, 1] = [D, E].\end{aligned}$$

Let us set $S_i = \sigma(U_i) \cup \{t_i\}$ and define for $s \in S_i$

$$f_i(s) = \begin{cases} \operatorname{re}(x_n(s)), & s \in \sigma_i(U) \\ \operatorname{re}(\operatorname{sgn}(y_i(s))), & s = t_i \end{cases}$$

and

$$g_i(s) = \begin{cases} \operatorname{im}(x_n(s)), & s \in \sigma(U_i) \\ \operatorname{im}(\operatorname{sgn}(y_i(s))), & s = t_i. \end{cases}$$

Following the Tietze–Urysohn Theorem, we can extend in a continuous way the functions f_i and g_i on the whole set $\bar{U}_i$ such that $f_i(s) \in [B, C]$ and $g_i\,(s) \in [D, E]$ for every $s \in \bar{U}_i$. It is easy to show that

$$|(f_i + i \cdot g_i)(s) - \operatorname{sgn}(y_i(t))| \leqslant 2 \cdot \sqrt{2} \cdot \varepsilon_n.$$

Let π_i: $B_d(\operatorname{sgn}(y_i(t)), \sqrt{2} \cdot 2 \cdot \varepsilon_n) \to B_d(\operatorname{sgn}(y_i(t)), \sqrt{2} \cdot 2 \cdot \varepsilon_n) \cap B_d(0, 1)$ ($B_d(x, r) = \{y \in \mathbb{C}\colon |x - y| \leqslant r\}$) be a continuous function with $\pi_i|_{B_d(\operatorname{sgn}(y_i(t)), r) \cap B_d(0,1)} = id$ ($r = \sqrt{2} \cdot 2 \cdot \varepsilon_n$). Put $z_i^n = \pi_i \circ (f_i + i \cdot g_i)$. We note that z_i^n is continuous, $z_i^n(t_i) = \operatorname{sgn}(y_i(t))$ and $\sup\{|z_i^n(s)|\colon s \in \bar{U}_i\} = 1$. Now define a function z_n: $T \to \mathbb{C}$ by

$$z_n(s) = \begin{cases} x_n(s)\colon s \in T \setminus \bigcup_{i \in A} \bar{U}_i \\ z_i^n(s)\colon s \in \bar{U}_i. \end{cases}$$

Since for every $i \in A$ and $s \in \sigma(U_i)$ $z_n(s) = x_n(s)$, z_n is continuous. Moreover $\|z_n\| = 1$ and $z_n(t_i) = \operatorname{sgn}(y_i(t))$ for $i \in A$, which means that $z_n \in A_t$.

To finish the proof, it is sufficient to show $\|z_n - x_n\| \to 0$. Fix $s \in T$. If $s \in T \setminus \bigcup_{i \in A} U_i$, then $|(x_n - z_n)(s)| = 0$. If $s \in \bar{U}_i$ for some $i \in A$, then $|x_n(s) - z_n(s)| \leqslant |x_n(s) - x_n(t_i)| + |x_n(t_i) - \operatorname{sgn}(y_i(t))| + |\operatorname{sgn}(y_i(t)) - z_n(s)| \leqslant (2 + \sqrt{2} \cdot 2) \cdot \varepsilon_n$. But this gives that $\|z_n - x_n\| \to 0$, since $\varepsilon_n \to 0$. The proof is completed.

Now we will prove the main result of this section.

THEOREM 3.2. *Let $\mathscr{V} \subset \mathscr{K}_F(X)$ be a convex set. Take $K \in \mathscr{K}_F(X)$, $V \in \mathscr{V}$, and assume $K - V \in \mathscr{D}(X)$. Then we have:*

(a) *$V \in P_{\mathscr{V}}(K)$ if and only if for every $U \in \mathscr{V}$ there exists $\hat{t} \in \operatorname{crit}(K-V)$ such that $\inf\{\operatorname{re}(((U-V)x)t)\colon x \in A_t\} \leqslant 0$, where A_t is defined by* (3.1).

(b) *V is a SUBA to K in $\mathscr{V}$ with a constant $r > 0$ if and only if for every $U \in \mathscr{V}$ there exists $\hat{t} \in \operatorname{crit}(K-V)$ such that $\inf\{\operatorname{re}(((U-V)x)t)\colon x \in A_t\} \leqslant -r \cdot \|U-V\|$.*

Proof. (a) Assume that $V \notin P_{\mathscr{V}}(K)$. Then there exists $U \in \mathscr{V}$ with $\|K-U\| < \|K-V\|$. Take $\hat{t} \in \operatorname{crit}(K-V)$ and $x \in A_t$. We note that

$$\begin{aligned}\operatorname{re}(((U-V)x)t) &= \operatorname{re}(((K-V)x)t) - \operatorname{re}(((K-U)x)t)\\ &\geqslant \|K-V\| - \|K-U\| > 0\end{aligned}$$

and consequently $\inf\{\operatorname{re}(((U-V)x)t)\colon x \in A_t\} > 0$.

To prove the converse suppose that for some $U \in \mathscr{V}$ and every $\hat{t} \in \operatorname{crit}(K-V)$ $\inf\{\operatorname{re}(((U-V)x)t)\colon x \in A_t\} > 0$. Following Theorem 0.1 it is sufficient to show that $\operatorname{re}(f(U-V)) > 0$ for every $f \in E(K-V)$ (see (0.1)). So fix $f \in E(K-V)$. By Theorem 1.2 and Corollary 1.3, $f = x^{**} \otimes \hat{t}$ for some $t \in T$ and $x^{**} \in \operatorname{ext} S(X^{**})$. Applying Goldstine's Theorem we may select a net $\{x_\beta\} \subset S(X)$ tending weak* in X^{**} to x^{**}. Following (1.1), we note that

$$\begin{aligned}\|K-V\| &\geqslant \operatorname{re}(((K-V)x_\beta)t) \to \operatorname{re}(((K-V)^* x^{**})t)\\ &= \operatorname{re}(f(K-V)) = \|K-V\|\end{aligned}$$

and consequently $\hat{t} \in \operatorname{crit}(K-V)$.

Now let us set $f_\beta = x_\beta \otimes \hat{t}$ and observe that for every $W \in \mathscr{K}(X)$

$$f_\beta(W) = \hat{t}(W(x_\beta)) \to \hat{t}(W^*(x^{**})) = (x^{**} \otimes \hat{t})(W^*) = f(W).$$

Hence we may select a sequence $\{f_n\} \subset \{f_\beta\}$ $(f_n = x_n \otimes \hat{t})$ such that $f_n(K-V) \to f(K-V) = \|K-V\|$ and $f_n(U-V) = ((U-V)x_n)t \to f(U-V)$. Following Lemma 3.1, there exists a sequence $\{z_n\} \subset A_t$ with $\|z_n - x_n\| \to 0$. It is clear that

$$((U-V)z_n - (U-V)x_n)t \to 0 \qquad \text{which yields } ((U-V)z_n)t \to f(U-V).$$

Since for $n = 1, 2, \ldots, z_n \in A_t$ and $t \in \operatorname{crit}(K-V)$, $\operatorname{re}(f(U-V)) > 0$ which according to Theorem 0.1 completes the proof of part (a). Applying Theorem 0.2, part (b) can be shown in the same way.

Remark 3.3. Assume $\mathscr{V}$, $\mathscr{K}_F(X)$, K, V are the same as in Theorem 3.2.

Assume furthermore that $\operatorname{card}(F) < +\infty$. Then the assumption $K - V \in \mathscr{D}(X)$ is superfluous and Theorem 3.2 yields a complete Kolmogorov's type characterization of best approximants and SUBA elements in this case.

Theorem 3.2 yields immediately the following result:

THEOREM 3.4. *Let $Y \subset X$ be its n-dimensional subspace and let $\mathscr{V} = \mathscr{P}(X, Y)$ (see Preliminaries). Assume $P_0 \in \mathscr{P}(X, Y) \cap \mathscr{D}(X, Y)$. Then P_0 is minimal in $\mathscr{P}(X, Y)$ (resp. P_0 is a SUBA to 0 in $\mathscr{P}(X, Y)$ with a constant $r > 0$) if and only if for every $P \in \mathscr{P}(X, Y)$ there exists $\hat{t} \in \operatorname{crit}(P_0)$ such that $\inf\{\operatorname{re}(((P_0 - P)x)t)\colon x \in A_t\} \leqslant 0$ (resp. $\leqslant -r \cdot \|P - P_0\|$ in the case of strong unicity).*

Proof. Take $K = 0$, $V = P_0$, and note that $\operatorname{crit}(P_0) = \operatorname{crit}(-P_0)$. By Theorem 3.2, we derive the desired result.

We note that Theorem 3.4 extends the result of Cheney (see [4]) proved for $P_0 \in I(X, Y)$ (see (1.8)) in the real case.

Now we apply Theorem 3.4 to generalize the other well known theorem from the theory of minimal projections. At first we introduce some notions.

Let $Y \subset X$, $\dim(Y) = n$, and let $F = \{t_1, ..., t_m\}$, $t_i \neq t_j$ for $i \neq j$, $m \geqslant n + 1$. Assume furthermore that F is total over Y, i.e., if $y \in Y$, $y(t_j) = 0$ for $j = 1, ..., m$, then $y = 0$. Since $\dim(Y) = n$, we may numerate the points from F in such a way that $(\hat{t}_1|_Y, ..., \hat{t}_n|_Y)$ form a basis of Y^*. For $i = n+1, ..., m$ put $B_i = \{1, ..., n, i\}$ and select for $j \in B_i$ the numbers τ_i^j such that $\sum_{j \in B_i} |\tau_i^j| > 0$ and $\sum_{j \in B_i} (\tau_i^j \cdot \hat{t}_j)|_Y = 0$.

Let us assume $P \in \mathscr{P}(X, Y, F)$ (see Preliminaries), $P = \sum_{j=1}^m \hat{t}_j(\cdot) \cdot u_j$, where $u_j \in Y$ for $j = 1, ..., m$. For $i = n+1, ..., m$ define the functions $v_i^P\colon T \to \mathbb{C}$ by

$$v_i^P(s) = \sum_{j \in B_i} \tau_i^j \cdot \operatorname{sgn}(u_j(s)) \tag{3.2}$$

and the functionals ϕ_i by

$$\phi_i = \sum_{j \in B_i} \tau_i^j \cdot \hat{t}_j. \tag{3.3}$$

Then we can prove the following

THEOREM 3.5. (a) *P is not a minimal projection in $\mathscr{P}(X, Y, F)$ if and only if for every $i \in \{n+1, ..., m\}$ there exists $y_i \in Y$ such that for every $\hat{s} \in \operatorname{crit}(P)$*

$$\operatorname{re}\left(\sum_{i=n+1}^{m} v_i^P(s) \cdot y_i(s) - \sum_{j \in B_s^P} \left|\sum_{i=n+1}^{m} \tau_i^j \cdot y_i(s)\right| - \sum_{j \in C_s^P} |y_j(s) \cdot \tau_j^j|\right) > 0, \tag{3.4}$$

where $B_s^P = \{j \in \{1, ..., n\}: u_j(s) = 0\}$, $C_s^P = \{j \in \{n+1, ..., m\}: u_j(s) = 0\}$, $\sum_{j \in B_s^P} = 0$ *(resp.* $\sum_{j \in C_s^P} = 0$*) if* $B_s^P = \emptyset$ *(resp.* $C_s^P = \emptyset$*).*

(b) *P is not a SUBA to 0 in* $\mathscr{P}(X, Y, F)$ *with a constant* $r > 0$ *if and only if for every* $i = n+1, ..., m$ *there exists* $y_i \in Y$ *such that for every* $\hat{s} \in \operatorname{crit}(P)$

$$\operatorname{re}\left(\sum_{i=n+1}^{m} v_i^P(s) \cdot y_i(s) - \sum_{j \in B_s^P} \left|\sum_{i=n+1}^{m} \tau_i^j \cdot y_i(s)\right| - \sum_{j \in C_s^P} |\tau_j^j \cdot y_j(s)|\right) > -r \cdot \|L\|, \tag{3.5}$$

where $L = \sum_{i=n+1}^{m} \phi_i(\cdot) \cdot y_i$.

Proof. (a) Assume that condition (3.4) is fulfilled and let $L = \sum_{i=n+1}^{m} \phi_i(\cdot) \cdot y_i$. To prove that P is not a minimal projection, in view of Theorem 3.4, it is sufficient to show that for each $\hat{s} \in \operatorname{crit}(P)$

$$\inf\{\operatorname{re}((Lx)s): x \in A_s\} > 0.$$

Let us denote for $i = n+1, ..., m$, $D_i = \{j \in B_i: u_j(s) \neq 0\}$ and $E_i = B_i \setminus D_i$. Fix $s \in \operatorname{crit}(P)$, $x \in A_s$, and compute

$$\begin{aligned}(Lx)s &= \sum_{i=n+1}^{m} \phi_i(x) \cdot y_i = \sum_{i=n+1}^{m} \left(\sum_{j=1}^{n} \tau_i^j \cdot x(t_j) + \tau_i^i \cdot x(t_i)\right) \cdot y_i \\ &= \sum_{i=n+1}^{m} \left(\sum_{j \in D_i} \tau_i^j \cdot \operatorname{sgn}(u_j(s)) - \sum_{j \in E_i} \tau_i^j \cdot (-x(t_j))\right) \cdot y_i \\ &= \sum_{i=n+1}^{m} v_i^P(s) \cdot y_i(s) - \sum_{i=n+1}^{m} \left(\sum_{j \in E_i} \tau_i^j \cdot (-x(t_j)) \cdot y_i(s)\right) \\ &= \sum_{i=n+1}^{m} v_i^P(s) \cdot y_i(s) - \sum_{j \in B_s^P} \left(\sum_{i=n+1}^{m} \tau_i^j \cdot y_i(s)\right) \cdot (-x(t_j)) \\ &\quad - \sum_{j \in C_s^P} \tau_j^j \cdot y_j(s) \cdot (-x(t_j)).\end{aligned}$$

Consequently, since $\|x\| \leqslant 1$, we obtain

$$\begin{aligned}\operatorname{re}((Lx)s) \geqslant \operatorname{re}\Bigg(&\sum_{i=n+1}^{m} v_i^P(s) \cdot y_i(s) - \sum_{j \in B_s^P} \left|\sum_{i=n+1}^{m} \tau_i^j \cdot y_i(s)\right| \\ &- \sum_{j \in C_s^P} |\tau_j^j \cdot y_j(s)|\Bigg) > 0.\end{aligned}$$

By Theorem 3.4, P is not a minimal projection in $\mathscr{P}(X, Y, F)$.

To prove the converse, assume P is not minimal in $\mathscr{P}(X, Y, F)$ and choose $P_0 \in \mathscr{P}(X, Y, F)$ with $\|P_0\| < \|P\|$. By [6, Lemma 2], we may assume

$$P_0 = P + \sum_{i=n+1}^{m} \phi_i(\cdot) \cdot y_i \qquad \text{for some} \quad y_{n+1}, ..., y_m \in Y.$$

We show that the functions $y_{n+1}, ..., y_m$ satisfy (3.4). Fix $\hat{s} \in \text{crit}(P)$. By the Tietze–Urysohn Theorem we may define a function $x \in S(X)$ with the properties

$$x(t_j) = \begin{cases} \text{sgn}(u_j(s)), & u_j(s) \neq 0 \\ -\text{sgn}(\sum_{i=n+1}^{m} \tau_i^j \cdot y_i(s)), & u_j(s) = 0 \end{cases} \quad \text{for } j = 1, ..., n$$

and

$$x(t_j) = \begin{cases} \text{sgn}(u_j(s)), & u_j(s) \neq 0 \\ -\text{sgn}(\tau_j^j \cdot y_j(s)), & u_j(s) = 0 \end{cases} \quad \text{for } j = n+1, ..., m.$$

Observe that

$$(Px)s = \sum_{j=1}^{m} x(t_j) \cdot u_j(s) = \sum_{j \notin B_s^P \cup C_s^P} x(t_j) \cdot u_j(s) = \sum_{j=1}^{m} |u_j(s)| = \|P\|.$$

Calculating as in the previous part of the proof we obtain

$$((P_0 - P)x)s = \sum_{i=n+1}^{m} v_i^P(s) \cdot y_i(s) - \sum_{j \in B_s^P} \left| \sum_{i=n+1}^{m} \tau_i^j \cdot y_i(s) \right| - \sum_{j \in C_s^P} |\tau_j^j \cdot y_j(s)|.$$

Since, following Theorem 3.4, $\text{re}(((P_0 - P)x)s)) > 0$, the proof of part (a) is fully completed.

The proof of part (b) goes on the same line, so we omit it.

Observe that in the real case if $m = n + 1$ condition (3.4) reduces to

$$|y_{n+1}(s)| \cdot \left(v_{n+1}^P(s) \cdot \text{sgn}(y_{n+1}(s)) - \sum_{j \in B_s^P \cup C_s^P} |\tau_{n+1}^j| \right) > 0 \qquad (3.6)$$

which after dividing by $|y_{n+1}(s)|$ yields the result of Cheney (see [8, Theorem 5]).

4. The Case of Sequence Spaces

Assume $Y \subset c_0$ (see Preliminaries) is an n-dimensional subspace and let $y_1, ..., y_n$ be a basis of Y. For $K \in \mathscr{K}(c_0, Y)$, $K = \sum_{i=1}^n f_i(\cdot) \cdot y_i$ ($f_i \in l_1$ for $i = 1, ..., n$) put

$$K_K(s, t) = \sum_{i=1}^{n} f_i(s) \cdot y_i(t) \qquad \text{for} \quad s, t \in T. \tag{4.1}$$

As in the previous section for $t \in T$ the symbol $\hat{t}$ stands for the evaluation functional. By [9, Lemma 1] and (2.1),

$\hat{t} \in \operatorname{crit}(K)$ if and only if t is a critical point of the function Λ_K: $T \to \mathbb{R}_+$ defined by $\Lambda_K(s) = \|\sum_{i=1}^n y_i(s) \cdot f_i\|_1 = \sum_{u \in T} |K_K(u, s)|$, i.e., $\Lambda_K(t) = \sup\{\Lambda_K(s)\colon\ s \in T\}$ (the symbol $\|\cdot\|_1$ denotes the norm in the space l_1). (4.2)

Using these notations we may prove the following

THEOREM 4.1. *Let $\mathscr{V} \subset \mathscr{K}(c_0, Y)$ be a convex set and let $K \in \mathscr{K}(c_0, Y)$, $V \in \mathscr{V}$. Then we have:*

(a) *$V \in P_{\mathscr{V}}(K)$ if and only if for every $U \in \mathscr{V}$ there exists $\hat{t} \in \operatorname{crit}(K - V)$ with*

$$\operatorname{re}\left(\sum_{s \in T} K_{U-V}(s, t) \cdot \operatorname{sgn}(K_{K-V}(s, t))\right) - \sum_{s \in A_t} |K_{U-V}(s, t)| \leqslant 0. \tag{4.3}$$

(b) *$V \in \mathscr{V}$ is a SUBA to K in $\mathscr{V}$ with a constant $r > 0$ if and only if for every $U \in \mathscr{V}$ there exists $\hat{t} \in \operatorname{crit}(K - V)$ such that*

$$\operatorname{re}\left(\sum_{s \in T} K_{U-V}(s, t) \cdot \operatorname{sgn}(K_{K-V}(s, t))\right) - \sum_{s \in A_t} |K_{U-V}(s, t)| \leqslant -r \cdot \|U - V\|, \tag{4.4}$$

where $A_t = \{s \in T\colon K_{K-V}(s, t) = 0\}$.

Proof. Assume there exists $U \in \mathscr{V}$ such that for every $\hat{t} \in \operatorname{crit}(K - V)$, (4.3) does not hold. In view of Theorem 0.1, it is sufficient to show that $\operatorname{re}(\phi(U - V)) > 0$ for every $\phi \in E(K - V)$ (see (0.2)). Since $\mathscr{K}(c_0, Y) \subset \mathscr{K}(c_0)$, by Theorem 1.2 and Corollary 1.3, $\phi = \psi \otimes \gamma$ for some $\psi \in \operatorname{ext} S(c_0^{**})$ and $\gamma \in \operatorname{ext} S(c_0^*)$. Applying (1.4) and (1.5), we may assume that $\psi \in l_\infty(T)$, $|\psi(s)| = 1$ for every $s \in T$ and $\gamma = \hat{t}$ for some $t \in T$. Let

$K-V=\sum_{i=1}^n f_i(\cdot)\cdot y_i$ and $U-V=\sum_{i=1}^n g_i(\cdot)\cdot y_i$ for some f_i, $g_i\in l_1$. Following Remark 1.4 and (4.2) we note that

$$\begin{aligned}\|K-V\| &= \phi(K-V)=\hat{t}(K-V)^*\psi=\sum_{i=1}^n \psi(f_i)\cdot y_i(t)\\ &= \sum_{i=1}^n \left(\sum_{s\in T} f_i(s)\cdot\psi(s)\right)\cdot y_i(t)\\ &= \sum_{s\in T}\psi(s)\cdot\left(\sum_{i=1}^n f_i(s)\cdot y_i(t)\right)\\ &\leqslant \sum_{s\in T}|K_{K-V}(s,t)|=\|K-V\|.\end{aligned}$$

It means that $\psi(s)=\operatorname{sgn}(K_{K-V}(s,t))$ if $s\in T\backslash A_t$. Compute

$$\begin{aligned}\operatorname{re}(\phi(U-V)) &= \operatorname{re}\left(\sum_{i=1}^n \psi(g_i)\cdot y_i(t)\right)\\ &= \operatorname{re}\left(\sum_{i=1}^n\left(\sum_{s\in T}\psi(s)\cdot g_i(s)\right)\cdot y_i(t)\right)\\ &= \operatorname{re}\left(\sum_{s\in T}\psi(s)\cdot\left(\sum_{i=1}^n g_i(s)\cdot y_i(t)\right)\right)\\ &= \operatorname{re}\left(\sum_{s\in T}\psi(s)\cdot K_{U-V}(s,t)\right)\\ &= \operatorname{re}\left(\sum_{s\in T} K_{U-V}(s,t)\cdot\operatorname{sgn}(K_{K-V}(s,t))\right.\\ &\qquad\left.-\sum_{s\in A_t}(-\psi(s))\cdot K_{U-V}(s,t)\right).\end{aligned}$$

Since $|\operatorname{re}(\sum_{s\in A_t}(-\psi(s)\cdot K_{U-V}(s,t)))|\leqslant\sum_{s\in A_t}|K_{U-V}(s,t)|$, $\operatorname{re}(\phi(U-V))\geqslant\operatorname{re}(\sum_{s\in T}K_{U-V}(s,t)\cdot\operatorname{sgn}(K_{K-V}(s,t)))-\sum_{s\in A_t}|K_{U-V}(s,t)|>0$. Following Theorem 0.1, $V\notin P_{\mathscr{V}}(K)$.

To prove the converse, suppose $V\notin P_{\mathscr{V}}(K)$ and choose $U\in\mathscr{V}$ with $\|U-K\|<\|V-K\|$. Let $\hat{t}\in\operatorname{crit}(K-U)$ be fixed. Define a function $\psi\in l_\infty$ by

$$\psi(s)=\begin{cases}\operatorname{sgn}(K_{K-V}(s,t)), & K_{K-V}(s,t)\neq 0\\ -\operatorname{sgn}(K_{U-V}(s,t)), & K_{K-V}(s,t)=0,\ K_{U-V}(s,t)\neq 0\\ 1, & \text{in the opposite case.}\end{cases}$$

Let us set $\phi = \psi \otimes \hat{t}$. Following [16], $\phi \in \operatorname{ext} S(\mathscr{K}(c_0))$. Observe that

$$\phi(K-V) = \sum_{i=1}^{n} \psi(f_i) \cdot y_i(t) = \sum_{i=1}^{n} \left(\sum_{s \in T} \psi(s) \cdot f_i(s) \right) y_i(t)$$

$$= \sum_{s \in T} \psi(s) \cdot \left(\sum_{i=1}^{n} f_i(s) \cdot y_i(t) \right)$$

$$= \sum_{s \in T} |K_{K-V}(s, t)| = \|K - V\|.$$

Hence $\phi \in E(K-V)$ and, by Theorem 0.1, $\operatorname{re}(\phi(U-V)) > 0$. But

$$\operatorname{re}(\phi(U-V)) = \operatorname{re}\left(\sum_{s \in T} \psi(s) \cdot K_{U-V}(s, t) \right)$$

$$= \operatorname{re}\left(\sum_{s \in T} K_{U-V}(s, t) \cdot \operatorname{sgn}(K_{K-V}(s, t)) \right) - \sum_{s \in A_t} |K_{U-V}(s, t)|,$$

which gives the desired result.

Following Theorem 0.2, part (b) can be proved in the same way.

Remark 4.2. In the real case for $K = 0$ and $\mathscr{V} = \mathscr{P}(c_0, Y)$ Theorem 4.1(a) was proved by a different method in [9, Theorem 1].

Now we present a similar result for the space $\mathscr{K}(l_1, Y)$. To do this, for $K \in \mathscr{K}(l_1, Y)$, $K = \sum_{i=1}^{n} f_i(\cdot) \cdot y_i$, where $f_i \in l_\infty$ for $i = 1, ..., n$ and $y_1, ..., y_n$ is a fixed basis of Y, put

$$K_K(\psi, t) = \sum_{i=1}^{n} \psi(f_i) \cdot y_i(t), \qquad \psi \in l_1^{**}, t \in T.$$

Following the Banach–Alaoghlu and the Krein–Milman Theorems, and by the definition of the space $\mathscr{L}_e(l_1^{**}, Y)$ (see Proposition 1.1), we note that the set

$$C_K = \{\psi \in \operatorname{ext}(S(l_1^{**})) \colon K^*(\psi) = \|K\|\} \tag{4.5}$$

is nonvoid. Moreover

$$\psi \in C_K \text{ if and only if } \sum_{t \in T} K_K(\psi, t) = \|K\|. \tag{4.6}$$

Using the above notations we can prove the following

THEOREM 4.3. *Let $\mathscr{V} \subset \mathscr{K}(l_1, Y)$ be a convex set and let $K \in \mathscr{K}(l_1, Y)$, $V \in \mathscr{V}$. Then we have*:

(a) *$V \in P_{\mathscr{V}}(K)$ if and only if for every $U \in \mathscr{V}$ there exists $\psi \in C_{K-V}$ such that*

$$\operatorname{re}\left(\sum_{t\in T} K_{U-V}(\psi, t)\cdot \operatorname{sgn}(K_{K-V}(\psi, t)) - \sum_{t\in A_\psi} |K_{U-V}(\psi, t)|\right) \leqslant 0. \tag{4.7}$$

(b) *V is a SUBA to K in $\mathscr{V}$ with a constant $r>0$ if and only if for every $U\in\mathscr{V}$ there exists $\psi\in C_{K-V}$ with*

$$\operatorname{re}\left(\sum_{t\in T} K_{U-V}(\psi, t)\cdot \operatorname{sgn}(K_{K-V}(\psi, t)) - \sum_{t\in A_\psi} |K_{U-V}(\psi, t)|\right)$$
$$\leqslant -r\cdot\|U-V\|, \tag{4.8}$$

where $A_\psi = \{t\in T\colon K_{K-V}(\psi, t)=0\}$.

Proof. (a) Fix $K\in\mathscr{K}(l_1, Y)$ and $V\in P_{\mathscr{V}}(K)$. Let $K-V=\sum_{i=1}^n f_i(\cdot)\cdot y_i$. Assume that for some $U\in\mathscr{V}$, (4.7) is not fulfilled. Suppose $U-V=\sum_{i=1}^n g_i(\cdot)\cdot y_i$ and take $\phi\in E(K-V)$. We show that $\operatorname{re}(\phi(U-V))>0$. To do this, we note that following Theorem 1.2 and Corollary 1.3, $\phi=\psi\otimes\gamma$, where $\psi\in\operatorname{ext} S(l_1^{**})$ and $\gamma\in\operatorname{ext} S(l_1^*)$. By (1.5), we may assume that $\gamma\in S(l_\infty)$ and $|\gamma(t)|=1$ for every $t\in T$. Observe that

$$\|K-V\| = \phi(K-V) = \gamma((K-V)^*\,\psi)$$
$$= \gamma\left(\sum_{i=1}^n \psi(f_i)\cdot y_i\right) = \sum_{t\in T}\gamma(t)\cdot K_{K-V}(\psi, t)$$
$$\leqslant \sum_{t\in T} |K_{K-V}(\psi, t)| \leqslant \|K-V\|.$$

By (4.6), $\psi\in C_{K-V}$. Hence $\gamma(t)=\operatorname{sgn}(K_{K-V}(\psi, t))$ if $t\in T\backslash A_\psi$. Compute

$$\operatorname{re}(\phi(U-V)) = \operatorname{re}\left(\gamma\left(\sum_{i=1}^n \psi(g_i)\cdot y_i\right)\right)$$
$$= \operatorname{re}\left(\sum_{t\in T}\gamma(t)\cdot K_{U-V}(\psi, t)\right)$$
$$= \operatorname{re}\left(\sum_{t\in T} K_{U-V}(\psi, t)\cdot\operatorname{sgn}(K_{K-V}(\psi, t)) - \sum_{t\in A_\psi} K_{U-V}(\psi, t)\cdot(-\gamma(t))\right)$$
$$\geqslant \operatorname{re}\left(\sum_{t\in T} K_{U-V}(\psi, t)\cdot\operatorname{sgn}(K_{K-V}(\psi, t)) - \sum_{t\in A_\psi} |K_{U-V}(\psi, t)|\right) > 0.$$

By Theorem 0.1, $V\notin P_{\mathscr{V}}(K)$.

Now suppose $V \notin P_{\mathscr{V}}(K)$ and take $U \in \mathscr{V}$ with $\|K-U\| < \|K-V\|$. Choose $\psi \in C_{K-V}$ and define $\gamma \in \text{ext}(S(l_\infty))$ by

$$\gamma(t) = \begin{cases} \text{sgn}(K_{K-V}(\psi, t)), & K_{K-V}(\psi, t) \neq 0 \\ -\text{sgn}(K_{U-V}(\psi, t)), & K_{K-V}(\psi, t) = 0 \text{ and } K_{U-V}(\psi, t) \neq 0 \\ 1, & \text{in the opposite case.} \end{cases}$$

Let $\phi = \psi \otimes \gamma$. Following [16], $\phi \in \text{ext}\, S(K(l_1))$. Observe that, by Remark 1.4 and (4.6),

$$\begin{aligned} \phi(K-V) &= \gamma\left(\sum_{i=1}^{n} \psi(f_i) \cdot y_i\right) \\ &= \sum_{t \in T} \gamma(t) \cdot K_{K-V}(\psi, t) = \sum_{t \in T \setminus A_\psi} |K_{K-V}(\psi, t)| \\ &= \|K-V\|. \end{aligned}$$

Hence, by Theorem 0.1, $\text{re}(\phi(U-V)) > 0$. But

$$\begin{aligned} \text{re}(\phi(U-V)) &= \text{re}\left(\sum_{t \in T} \gamma(t) \cdot K_{U-V}(\psi, t)\right) \\ &= \text{re}\left(\sum_{t \in T} K_{U-V}(\psi, t) \right. \\ &\qquad \left. \cdot\, \text{sgn}(K_{K-V}(\psi, t)) - \sum_{t \in A_\psi} |K_{U-V}(\psi, t)|\right), \end{aligned}$$

which gives the desired result.

By (1.6), Theorem (4) of [9], and similar reasoning as in Theorem 4.3, we can prove the following

THEOREM 4.4. *Let $\mathscr{V} = \mathscr{P}(l_1, Y)$ and $K = 0$. Assume furthermore that $\dim(Y|_A) = \dim(Y)$ for every infinite set $A \subset \{t \in T\colon y(t) \neq 0$ for some $y \in Y\}$. Then $V \in \mathscr{V}$ is a minimal projection (resp. a SUBA to 0 in $\mathscr{V}$) if and only if Theorem* 4.3(a) (*resp. Theorem* 4.3(b)) *holds true with $\psi \in C_{K-V}$ replacing $\hat{s} \in C_{K-V}$, where $s \in T$.*

The above criterion for minimal projections in the real case has been proved (by a different method) in [9, Theorem 5].

Note added in the proof. It is clear, by Theorems 0.1 and 0.2, that Theorems 2.2, 2.3, 3.2, 4.1, 4.3 and Corollary 3.3 hold true under the weaker assumptions on the set $\mathscr{V}$.

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